chosen to correspond, in particular, to the conditions of applicability of the equations used. If the flow is stable, then the perturbations will not increase and the conditions indicated will also be observed at $t > 0$. Thus the arguments concerning the stability in its classical sense made on the basis of utilizing Eqs. (1.1) or (1.8) are valid outside the range of their dependence on the magnitude of the difference $U - A$, provided that the latter does not vanish at $0 < x < 1$.

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NONLINEAR PROPAGATION OF WAVES IN MEDIA WITH AN ARBITRARY NUMBER OF CHEMICAL REACTIONS

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We consider nonlinear wave motions in chemically active, gaseous mixtures the change in the composition of which is governed by an arbitrary number of reactions taking place. We impose on the equations of state the conditions ensuring that the frozen and the equilibrium speed of sound have similar values. We carry out an asymptotic analysis of the initial system of Euler equations together with the chemical reaction equations. As the result, we obtain an approximate system of equations for the velocity of the medium particles, and for the reaction completeness vector the order of which is equal to the number of the relaxation processes plus one.

1. Thermodynamics of the system. We assume that N reactions take place in the flow of a chemically active gaseous mixture. The change in the composition of the mixture is characterized by the vector $\mathbf{q} = (q_1, \ldots, q_N)$, which we shall call the reaction completeness vector. The concentrations of the components depend on the magnitude of this vector. Let us denote by t the time, by r the distance from the plane, axis or center of symmetry, by v the particle velocity, ρ the density, p the pressure and let $\mathbf{q}^{\cdot} = (q_1^{\cdot}, \ldots, q_N^{\cdot})$ and $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_N)$ be the vectors defining the rate and the affinity of the chemical reactions. The equations of motion of the medium are written in the form [1, 2]

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho v}{\partial r} + (\nu - 1)\frac{\rho v}{r} = 0, \quad \rho\frac{\partial v}{\partial t} + \rho v\frac{\partial v}{\partial r} + \frac{\partial p}{\partial r} = 0 \tag{1.1}$$

$$\boldsymbol{\omega}\left(\frac{\partial\mathbf{q}}{\partial t} + v\frac{\partial\mathbf{q}}{\partial r}\right) + T\left(\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial r}\right) = 0$$

$$\frac{\partial\mathbf{q}}{\partial t} + v\frac{\partial\mathbf{q}}{\partial r} = \mathbf{q}^{\cdot}$$

where the values of the parameter $\nu = 1, 2, 3$ correspond to the flows with a plane, axis and center of symmetry, respectively.

To close the above system, we must supplement it with another $N + 2$ equations connecting the thermodynamic functions $\mathbf{q}$, $\boldsymbol{\omega}$, ρ, p, s and T. In accordance with the Gibbs relation the increment in the specific internal energy e, is

$$de = \boldsymbol{\omega} d\mathbf{q} - pdV + Tds, \quad V = 1/\rho$$

The first partial derivatives

$$\omega_i = \left(\frac{\partial e}{\partial q_i}\right)_{q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_N, V, s} = \left(\frac{\partial e}{\partial q_i}\right)_{q_j, V, s}$$

$$-p = \left(\frac{\partial e}{\partial V}\right)_{q_1, \ldots, q_N, s} = \left(\frac{\partial e}{\partial V}\right)_{q_j, s}$$

$$T = \left(\frac{\partial e}{\partial s}\right)_{q_1, \ldots, q_N, s} = \left(\frac{\partial e}{\partial s}\right)_{q_j, V}, \quad i, j = 1, \ldots, N$$

expressed in terms of $\mathbf{q}$, V and s, represent the equations of state of the medium, and provide the required $N + 2$ relations connecting the thermodynamic quantities.

As we know [1, 2], in the state of equilibrium the vector $\boldsymbol{\omega} = 0$. The chemical reaction rate vector $\mathbf{q}^{\cdot}$ vanishes as well. Let us choose $\boldsymbol{\omega}$, V and s as the independent thermodynamic variables. If we assume that the vector $\mathbf{q}^{\cdot}$ is analytically dependent on $\boldsymbol{\omega}$, then in the vicinity of the state of equilibrium we have

$$\mathbf{q}^{\cdot} = -\mathbf{H}(V, s)\boldsymbol{\omega} + \cdots \tag{1.2}$$

By virtue of the Onsager equivalence principle the matrix $\mathbf{H} = \| h_{il} \|$ is symmetric. In accordance with the second law of thermodynamics, the entropy of the system can only increase when the process is irreversible. From this we conclude that the matrix $\mathbf{H}$ must also be positive-definite.

We shall regard the pressure p as a function of the density ρ, entropy s and of N such quantities belonging to the sequence $q_1, \ldots, q_N, \omega_1, \ldots, \omega_N$, which are not conjugate. Obviously, such a choice of the independent variables can be made in 2^N different ways. Let us introduce the M-tuply frozen and $(N - M)$-tuply equilibrium speed of sound

$$a_{fe}^{(M)} = \left[\left(\frac{\partial p}{\partial \rho}\right)^{(M)}_{q_j,\,\omega_k,\,s}\right]^{1/2} \tag{1.3}$$

Here and henceforth the superscript accompanying the thermodynamic derivatives indicates that M components of the reaction completeness vector $\mathbf{q}$ and $N - M$ components of the vector $\boldsymbol{\omega}$ defining the chemical affinity, are referred to the independent variables. We have therefore $j = 1, \ldots, M$; $k = M + 1, \ldots, N$. We shall write, for simplicity

$$\left(\frac{\partial}{\partial q_m}\right)^{(N)}_{q_j,\,\omega_k,\,\rho,\,s} = \left(\frac{\partial}{\partial q_m}\right)_{q_j,\,\rho,\,s}, \quad \left(\frac{\partial}{\partial \omega_n}\right)^{(0)}_{q_j,\,\omega_k,\,\rho,\,s} = \left(\frac{\partial}{\partial \omega_n}\right)_{\omega_k,\,\rho,\,s}$$

where, in addition to the density and entropy, we use all components of either $\mathbf{q}$ or $\boldsymbol{\omega}$ as the independent thermodynamic variables. The completely frozen a_f and completely in equilibrium a_e speeds of sound are given by

$$a_f = a_{fe}^{(N)} = \left[\left(\frac{\partial p}{\partial \rho}\right)_{q_j,\,s}\right]^{1/2}, \quad a_e = a_{fe}^{(0)} = \left[\left(\frac{\partial p}{\partial \rho}\right)_{\omega_k,\,s}\right]^{1/2}$$

Differentiating with respect to ρ the identity

$$\begin{aligned} p(q_1, \ldots, q_M, \omega_{M+1}, \ldots, \omega_N;\ \rho, s) = p\,[q_1, \ldots, q_M,\ & \\ q_{M+1}(q_1, \ldots, q_M, \omega_{M+1}, \ldots, \omega_N, \rho, s), \ldots, q_N(q_1, \ldots, q_M, \omega_{M+1}, \ldots & \\ \omega_N, \rho, s); \rho, s] & \end{aligned} \tag{1.4}$$

we obtain

$$(a_{fe}^{(M)})^2 = a_f^2 + \sum_{n=M+1}^{N} \left(\frac{\partial p}{\partial q_n}\right)_{q_j,\,\rho,\,s} \left(\frac{\partial q_n}{\partial \rho}\right)^{(M)}_{q_j,\,\omega_k,\,s}$$

To transform the latter expression into a symmetrical form, we set

$$\psi_M = e - \sum_{n=M+1}^{N} \omega_n q_n$$

The following relation is valid for the increment in the function ψ_M:

$$d\psi_M = \sum_{m=1}^{M} \omega_m dq_m - \sum_{n=M+1}^{N} q_n d\omega_n - p dV + T ds$$

From this it follows that the derivative

$$\left(\frac{\partial q_n}{\partial \rho}\right)^{(M)}_{q_j,\,\omega_k,\,s} = -\frac{1}{\rho^2}\left(\frac{\partial q_n}{\partial V}\right)^{(M)}_{q_j,\,\omega_k,\,s} = -\frac{1}{\rho^2}\left(\frac{\partial p}{\partial \omega_n}\right)^{(M)}_{q_j,\,\omega_k,\,V,\,s}$$

Differentiating the identity (1.4) with respect to ω_n we obtain

$$\left(\frac{\partial p}{\partial \omega_n}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} = \sum_{l=M+1}^{N} \left(\frac{\partial p}{\partial q_l}\right)_{q_j,\,\rho,\,s} \left(\frac{\partial q_l}{\partial \omega_n}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} \tag{1.5}$$

Collecting the formulas obtained, we derive the following required relation:

$$a_f^2 - (a_{fe}^{(M)})^2 = \frac{1}{\rho^2} \sum_{l,\,n=M+1}^{N} \left(\frac{\partial q_l}{\partial \omega_n}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} \left(\frac{\partial p}{\partial q_l}\right)_{q_j,\,\rho,\,s} \left(\frac{\partial p}{\partial q_n}\right)_{q_j,\,\rho,\,s} \tag{1.6}$$

Taking into account the method used to introduce the function ψ_M, we can show that the matrix $\| (\partial q_l \,/\, \partial \omega_n)^{(M)}_{q_j,\omega_k,\rho,s} \|$ is symmetric. In fact

$$\left(\frac{\partial q_l}{\partial \omega_n}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} = -\left(\frac{\partial^2 \psi_M}{\partial \omega_n \partial \omega_l}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} = \left(\frac{\partial q_n}{\partial \omega_l}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s}$$

in the particular case at $M = 0$, the formula (1. 6) yields the expression

$$a_f^2 - a_e^2 = \frac{1}{\rho^2} \sum_{l,\,n=1}^{N} \left(\frac{\partial q_l}{\partial \omega_n}\right)_{\omega_k,\,\rho,\,s} \left(\frac{\partial p}{\partial q_l}\right)_{q_j,\,\rho,\,s} \left(\frac{\partial p}{\partial q_n}\right)_{q_j,\,\rho,\,s} \qquad (1.\,7)$$

for the difference of the squares of the frozen and of the equilibrium speed of sound. The elements of the matrix $\| (\partial q_l \,/\, \partial \omega_n)_{\omega_k,\rho,s} \|$ are defined in terms of the elements $\| (\partial \omega_l \,/\, \partial q_n)_{q_j,\rho,s} \|$, where the indices $l,\ n = 1, \ldots, N$. The second of these two matrices can be conveniently regarded as the straight matrix, in which case the first one will be an inverse matrix. Since

$$\left(\frac{\partial \omega_l}{\partial q_n}\right)_{q_j,\,\rho,\,s} = \left(\frac{\partial^2 e}{\partial q_n \partial q_l}\right)_{q_j,\,\rho,\,s} = \left(\frac{\partial \omega_n}{\partial q_l}\right)_{q_j,\,\rho,\,s}$$

the straight matrix $\| (\partial \omega_l \,/\, \partial q_n)_{q_j,\rho,s} \|$ will be symmetric. Moreover, the condition of thermodynamic stability of the system implies its positive definiteness [1, 2]. The symmetry of the inverse matrix $\| (\partial q_l \,/\, \partial \omega_n)_{\omega_k,\rho,s}$ follows directly from the above arguments, provided that we set $M = 0$. This enables us to conclude that the latter matrix is also positive definite.

Turning now our attention to the right-hand side of (1. 7), we find that the properties of the matrices deduced above, imply that

$$a_f^2 - a_e^2 \geqslant 0 \qquad (1.\,8)$$

The equality in (1. 8) is attained under the condition that all derivatives

$$(\partial p \,/\, \partial \omega_n)_{q_j,\rho,s} = 0, \quad n = 1, \ldots, N.$$

Since

$$\left(\frac{\partial \omega_n}{\partial \rho}\right)_{q_j,\,s} = -\frac{1}{\rho^2}\left(\frac{\partial \omega_n}{\partial V}\right)_{q_j,\,s} = \frac{1}{\rho^2}\left(\frac{\partial p}{\partial q_n}\right)_{q_j,\,\rho,\,s}$$

which means that either the pressure depends on the density and specific entropy only, or (which is equivalent) the vector ω defining the chemical affinity is a function of the reaction completeness vector and specific entropy only, but remains constant when the density changes.

2. Energy equation. Let us transform the third equation of (1. 1) which follows from the law of conservation of energy. We write the specific entropy increment in terms of the pressure and density increments, M components of the reaction completeness vector and $N - M$ components of the chemical affinity vector

$$ds = \left(\frac{\partial s}{\partial p}\right)^{(M)}_{q_j,\,\omega_k,\,\rho} \Big[dp - (a_{fe}^{(M)})^2 d\rho - \qquad (2.\,1)$$

$$\sum_{m=1}^{M} \left(\frac{\partial p}{\partial q_m}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} dq_m - \sum_{n=M+1}^{N} \left(\frac{\partial p}{\partial \omega_n}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} d\omega_n \Big]$$

From (2. 1) we find at once

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial r} - (a_{fe}^{(M)})^2\left(\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial r}\right) = L_{fe}^{(M)} = \tag{2.2}$$

$$-\frac{1}{T}\left(\frac{\partial p}{\partial s}\right)^{(M)}_{q_j,\,\omega_k,\,\rho}\sum_{i=1}^{N}\omega_i\left(\frac{\partial q_i}{\partial t} + v\frac{\partial q_i}{\partial r}\right) +$$

$$\sum_{m=1}^{M}\left(\frac{\partial p}{\partial q_m}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s}\left(\frac{\partial q_m}{\partial t} + v\frac{\partial q_m}{\partial r}\right) +$$

$$\sum_{n=M+1}^{M}\left(\frac{\partial p}{\partial \omega_n}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s}\left(\frac{\partial \omega_n}{\partial t} + v\frac{\partial \omega_n}{\partial r}\right)$$

and combining the above relation with the equations of continuity and conservation of impulse, we obtain

$$\frac{\partial p}{\partial t} + (v + a_{fe}^{(M)})\frac{\partial p}{\partial r} + \rho a_{fe}^{(M)}\left[\frac{\partial v}{\partial t} + (v + a_{fe}^{(M)})\frac{\partial v}{\partial r} + \frac{\nu - 1}{r}va_{fe}^{(M)}\right] = L_{fe}^{(M)} \tag{2.3}$$

If we refer all components of the reaction completeness vector to the independent thermodynamic variables, then

$$\frac{\partial p}{\partial t} + (v + a_f)\frac{\partial p}{\partial r} + \rho a_f\left[\frac{\partial v}{\partial t} + (v + a_f)\frac{\partial v}{\partial r} + \frac{\nu - 1}{r}va_f\right] = L_f = \tag{2.4}$$

$$\sum_{i=1}^{N}\left[\left(\frac{\partial p}{\partial q_i}\right)_{q_j,\,\rho,\,s} - \frac{\omega_i}{T}\left(\frac{\partial p}{\partial s}\right)_{q_j,\,\rho}\right]\left(\frac{\partial q_i}{\partial t} + v\frac{\partial q_i}{\partial r}\right)$$

Equations (2.3) and (2.4) express the consequences of the law of conservation of energy in two different ways. Formally, they are analogous to the relation often used in the theory of flow of inert gases [3] and transform to this relation when

$$\omega_i = (\partial p / \partial q_m)^{(M)}_{q_j,\omega_k,\,\rho,\,s} = (\partial p\, /\, \partial \omega_n)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} = 0$$

In the latter case the operators $L_{fe}^{(M)}$ and L_f appearing in the right-hand sides vanish identically; moreover the M-tuple frozen speeds of sound coincide with the $(N - M)$-tuple equilibrium speeds of sound for any $M = 0, 1, \ldots, N$, and are equal to the unique velocity $a = (\partial p\, /\, \partial \rho)_s^{1/2}$ of propagation of small perturbations.

3. Speeds of sound. Let us assume that although the magnitudes of all M-tuply frozen and $(N - M)$-tuply equilibrium speeds of sound in a reacting medium differ from each other, the differences are small. Let the quiescent state through which a wave of small amplitude propagates, represent the state of complete thermodynamic equilibrium. Denoting the parameters of the unperturbed gas by the index zero, we set

$$\left(\frac{\partial p}{\partial q_{i0}}\right)_{q_j,\,\rho,\,s} = -\left(\frac{\partial^2 e}{\partial q_{i0}\partial V_0}\right)_{q_j,\,s} = -\,\varepsilon_a\,\frac{p_0}{q_{i0}}\,e'_{iV_0}, \quad i = 1, \ldots, N \tag{3.1}$$

where ε_a is a small positive parameter and e'_{iV_0} are dimensionless quantities of the order of unity. The nonlinear perturbation theory for media in which a single chemical reaction takes place, was based on the above assumptions and developed in [4 — 7]. From (1.6) we see the difference

$$a_{f0}^2 - (a_{fe0}^{(M)})^2 \sim (N - M)^2 {\varepsilon_a}^2$$

When the number of the independent relaxation processes is finite, this difference is of the order of ε_a^2. Thus, by imposing the restrictions (3.1) on the equations of state of the medium, we ensure that all unperturbed speeds of sound differ slightly.

Let us clarify how the magnitudes of these speeds change with varying density, the reaction completeness vector and the chemical affinity vector. From the definition (1.3) follows

$$\left(\frac{\partial a_{fe}^{(M)}}{\partial q_m}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} = \frac{1}{2a_{fe}^{(M)}}\frac{\partial}{\partial\rho}\left(\frac{\partial p}{\partial q_m}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s}, \quad m = 1, \ldots, M \tag{3.2}$$

Differentiating the identities (1.4) with respect to q_m, we obtain

$$\left(\frac{\partial p}{\partial q_m}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} = \left(\frac{\partial p}{\partial q_m}\right)_{q_j,\,\rho,\,s} + \sum_{l=M+1}^{N}\left(\frac{\partial q_l}{\partial q_m}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s}\left(\frac{\partial p}{\partial q_l}\right)_{q_j,\,\rho,\,s} \tag{3.3}$$

Applying the formulas (3.2) and (3.3) to the equilibrium state of the mixture and substituting into them the quantities $(\partial p \,/\, \partial q_{l0})_{qj,\,\rho,\,s}$, we arrive, in accordance with Eqs. (3.1), at the following estimate:

$$\left(\frac{\partial a_{fe}^{(M)}}{\partial q_{m0}}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} \sim \varepsilon_a, \quad m = 1, \ldots, M \tag{3.4}$$

Returning to the definition (1.3), we further have

$$\left(\frac{\partial a_{fe}^{(M)}}{\partial \omega_n}\right)_{q_j,\,\omega_k,\,\rho,\,s} = \frac{1}{2a_{fe}^{(M)}}\frac{\partial}{\partial\rho}\left(\frac{\partial p}{\partial \omega_n}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s}, \quad n = M+1, \ldots, N \tag{3.5}$$

The derivatives $(\partial p \,/\, \partial \omega_n)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s}$ appearing in the above expression are defined by the relations (1.5). Writing them down together with the relations (3.5) for a quiescent mixture, we see that

$$\left(\frac{\partial a_{fe}^{(M)}}{\partial \omega_{n0}}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} \sim \varepsilon_a, \quad n = M+1, \ldots, N \tag{3.6}$$

In other words, changes in the values of the reaction completeness vectors and the chemical affinity vectors do not appreciably affect the values of the M-tuply frozen and $(N - M)$-tuply equilibrium speeds of sound.

Finally, in accordance with the definition (1.3), we have

$$\left(\frac{\partial a_{fe}^{(M)}}{\partial \rho}\right)^{(M)}_{q_j,\,\omega_k,\,s} = \frac{a_{fe}^{(M)}}{\rho}\left(m_{fe}^{(M)} - 1\right), \quad m_{fe}^{(M)} = \frac{1}{2\rho^3\,(a_{fe}^{(M)})^2}\left(\frac{\partial^2 p}{\partial V^2}\right)^{(M)}_{q_j,\,\omega_k,\,s}$$

As we know [3], the second partial derivative of the pressure with respect to specific volume, determines the stability of the inert gas flows. In the present case we have

$$\left(\frac{\partial^2 p}{\partial V^2}\right)^{(M)}_{q_j,\,\omega_k,\,s} = \left(\frac{\partial^2 p}{q V^2}\right)_{q_j,\,s} + Q_1 + Q_2 \tag{3.7}$$

$$Q_1 = \sum_{l=M+1}^{N}\left(\frac{\partial q_l}{\partial V}\right)^{(M)}_{q_j,\,\omega_k,\,s}\left[2\left(\frac{\partial^2 p}{\partial V \partial q_l}\right)_{q_j,\,s} + \sum_{n=M+1}^{N}\left(\frac{\partial q_n}{\partial V}\right)^{(M)}_{q_j,\,\omega_k,\,s}\left(\frac{\partial^2 p}{\partial q_l \partial q_n}\right)_{q_j,\,V,\,s}\right]$$

$$Q_2 = \sum_{l=M+1}^{N}\left(\frac{\partial p}{\partial q_l}\right)_{q_j,\,V,\,s}\left[\left(\frac{\partial}{\partial V}\left(\frac{\partial q_l}{\partial V}\right)^{(M)}_{q_j,\,\omega_k,\,s}\right)_{q_j,\,s} + \right.$$

$$\sum_{n=M+1}^{N} \left(\frac{\partial q_n}{\partial V}\right)^{(M)}_{q_j,\,\omega_k,\,s} \left(\frac{\partial}{\partial q_n} \left(\frac{\partial q_l}{\partial V}\right)^{(M)}_{q_j,\,\omega_k,\,s}\right)_{q_j,\,V,\,s}\Bigg]$$

Since $(\partial q_l \,/\, \partial V)^{(M)}_{q_j,\omega_k,s} = (\partial p \,/\, \partial \omega_l)^{(M)}_{q_j,\omega_k,V,s}$, it is clear that the quantity $Q_{10} \sim \varepsilon_a{}^2$. The last term appearing in the right-hand side of the first formula of (3.7) can be estimated by computing the following expression:

$$\left(\frac{\partial}{\partial q_n} \left(\frac{\partial q_l}{\partial V}\right)^{(M)}_{q_j,\,\omega_k,\,s}\right)_{q_j,\,V,\,s} = \sum_{\nu=M+1}^{N} \left(\frac{\partial^2 p}{\partial\omega_l \partial\omega_\nu}\right)^{(M)}_{q_j,\,\omega_k,\,V,\,s} \left(\frac{\partial \omega_\nu}{\partial q_n}\right)_{q_j,\,V,\,s}$$

$$\left(\frac{\partial}{\partial V} \left(\frac{\partial q_l}{\partial V}\right)^{(M)}_{q_j,\,\omega_k,\,s}\right)_{q_j,\,s} = \left(\frac{\partial^2 p}{\partial V \partial \omega_l}\right)^{(M)}_{q_j,\,\omega_k,\,s} -$$

$$\sum_{\nu=M+1}^{N} \left(\frac{\partial^2 p}{\partial\omega_l \partial\omega_\nu}\right)^{(M)}_{q_j,\,\omega_k,\,V,\,s} \left(\frac{\partial p}{\partial q_\nu}\right)_{q_j,\,V,\,s}$$

from which we see that $Q_{20} \sim \varepsilon_a{}^2$. Collecting together the results obtained, we arrive at the estimate

$$m_{f0} - m_{fe0}^{(M)} \sim \varepsilon_a{}^2, \qquad m_f = \frac{1}{2\rho^3 a_f{}^2} \left(\frac{\partial^2 p}{\partial V^2}\right)_{q_j,\,s} \tag{3.8}$$

4. Asymptotic expansions. Let us assume that the magnitudes of all characteristic quantities of the gas mixture deviate little, at any instant of time and every point of space, from their corresponding quiescent values. As we know [1 — 3], the transmission of signals in a relaxing medium is accompanied by dispersion, and in the limiting cases the speed of this transmission coincides either with the frozen, or with the equilibrium speed of sound. For the media considered here both the above speeds of sound have nearly equal values. Clearly, the velocity a_0 of propagation of the acoustic waves will not be very different from the above values, since it is bounded by the inequalities $a_{e0} \leqslant a_0 \leqslant a_{f0}$. In the nonlinear theory of small perturbations it may turn out that $a_{f0} < a_0$, e.g. the shock wave propagates at the velocity which exceeds both the equilibrium and the frozen speed of sound. We shall nevertheless assume that the difference between these quantities a_0, a_{e0} and a_{f0} is small, and we set

$$a_0 - a_{fe0}^{(M)} = \varepsilon_a{}^2 \beta^{(M)} a_0 \tag{4.1}$$

in the limiting cases when $M = 0$ and $M = N$ we have, respectively,

$$a_0 - a_{e0} = \varepsilon_a{}^2 \beta_e a_0, \qquad a_0 - a_{f0} = \varepsilon_a{}^2 \beta_f a_0 \tag{4.2}$$

the constants $\beta^{(M)}$, β_f and β_e are obviously of the order of unity.

Let us introduce a moving coordinate system attached to the element of the propagating wave, and denote by L the characteristic length in this system. We assume that the flow of the relaxing mixture represents a short wave, i.e. the width of the region in which the perturbations are concentrated is small compared with the distances covered by the propagating wave. Let Δ be a small numerical parameter, then the independent variables become

$$t = \frac{L}{\Delta a_0} t', \qquad r = a_0 t + L r' \tag{4.3}$$

Concerning the perturbations in the values of density, pressure, temperature, entropy and all speeds of sound considered above, we assume that they have the same order as the velocity of the medium particles, the latter proportional to another independent small parameter ε. Passing to the dimensionless unknown functions, we have

$$v = \varepsilon a_0 v', \quad \rho = \rho_0 (1 + \varepsilon\rho'), \quad p = p_0 (1 + \varepsilon p') \tag{4.4}$$

$$T = T_0 (1 + \varepsilon T'), \quad s = s_0 (1 + \varepsilon s'), \quad a_{fe}^{(M)} = a_{fe0}^{(M)} [1 + \varepsilon (a_{fe}^{(M)})']$$

We replace the right-hand side of the last equation of (1. 1) the vector $\mathbf{q}$ defining the rate of the relaxation processes, with its expression given by (1. 2). Inspecting the resulting expression and taking into account the sequence of equations

$$\left(\frac{\partial \omega_i}{\partial V}\right)_{q_j,\, s} = \varepsilon_a \frac{p_0}{q_{i0}} e'_{iV_0}, \quad i = 1, \ldots, N$$

we arrive at the conclusion that the perturbed reaction completeness vector is proportional to the product of the small parameters ε. and ε_a, and the chemical affinity vector must also be proportional to this product. Thus we have

$$q_i = q_{i0} (1 + \varepsilon\varepsilon_a q_i'), \quad \omega_l = \varepsilon\varepsilon_a \frac{p_0}{q_{l0}\rho_0} \omega_l', \quad i, l = 1, \ldots, N \tag{4.5}$$

5. Analysis of the Euler equations. In deriving the asymptotic equations, we shall retain in all relations only the principal terms, neglecting the terms of higher order of smallness. We shall also omit the primes accompanying all dimensionless variables.

After the linearization, integration of the first two equations of the system (1. 1) yields the formulas

$$\rho = \frac{p_0}{\rho_0 a_0^2} p = v \tag{5.1}$$

The first of these equations expresses the fact that, within the approximation considered, the compression of the gas is reversible. The second formula states that when the composition of the reaction mixture is constant, the Riemann relation characterizing a plane sound impulse propagating through an inert gas [3] holds for the whole flow of a multicomponent reaction mixture.

Taking into account the formulas (4. 4) and (4. 5), we find from the third equation of (1. 1) that $s \sim \varepsilon\varepsilon_a^2$, i.e. that within the adopted accuracy

$$s = 0 \tag{5.2}$$

Let us write a general expression for the deviation of the pressure from its equilibrium value in a quiescent medium, using the density, entropy, M components of the reaction completeness vector and $N - M$ components of the chemical affinity vector as the independent thermodynamic variables. We have

$$p_0 p = (a_{fe0}^{(M)})^2 \rho_0 \rho + \left(\frac{\partial p}{\partial s_0}\right)^{(M)}_{q_j,\, \omega_k,\, \rho} s_0 s + \sum_{m=1}^{M} \left(\frac{\partial p}{\partial q_{m0}}\right)^{(M)}_{q_j,\, \omega_k,\, \rho,\, s} q_{m0} \varepsilon_a q_m +$$

$$\sum_{n=M+1}^{N} \left(\frac{\partial p}{\partial \omega_{n0}}\right)^{(M)}_{q_j,\, \omega_k,\, \rho,\, s} \frac{p_0}{q_{n0}\rho_0} \varepsilon_a \omega_n$$

Here the second term in the right-hand side vanishes by virtue of Eq. (5. 2). Further, passing from the M-tuply frozen and $(N - M)$-tuply equilibrium speed of sound $a_{fe0}^{(M)}$ to the velocity of wave propagation a_0 with the help of (4. 1), taking into account the estimates of the thermodynamic derivatives obtained above and neglecting the terms proportional to ε_a^2, we return to the first formula of (5. 1).

Next we turn our attention to the chemical reaction rate vector. We set

$$h_{il0} = \frac{q_{i0} q_{l0} \rho_0}{\tau_{il} p_0} \, h'_{il0}$$

Here τ_{il} has the meaning of the time of relaxation of the i-th element, which is governed by the magnitude of the chemical affinity of the l-th element, and h'_{il0} is a dimensionless quantity of the order of unity. Since the matrix $\| h_{il} \|$ is symmetric, it is expedient to choose $\tau_{il} = \tau_{li}$. Substituting now the formulas (4. 3) and (4. 5) into the last equation of (1. 1) in which the vector $\dot{q}$ is replaced by its expression (1. 2), we find that in the first approximation

$$\frac{\partial q_i}{\partial r} = \sum_{l=1}^{N} N_{il} h_{il0} \omega_l, \quad N_{il} = \frac{L}{a_0 \tau_{il}}, \quad i = 1, \ldots, N \qquad (5.3)$$

We see that Eqs. (5. 3) contain additional numerical parameters N_{il} which express the rarios of the macroscopic times $\tau = L / a_0$ to the relaxation times τ_{il}. When all these ratios tend to zero or infinity, the analysis is considerably simplified. In the first case the thermodynamic state of the flow is almost frozen, and in the second case it is almost equilibrium.

The components of the chemical affinity vector can be expanded as follows:

$$\omega_i = - \; e_i V 0 \rho + \frac{\rho_0}{p_0} \sum_{i=1}^{N} \left(\frac{\partial \omega_i}{\partial q_{l0}} \right)_{q_j, V, s} q_{i0} q_{l0} q_l \qquad (5.4)$$

The components of the dimensionless relaxation processes rate vector can be defined as follows:

$$q_i^{\cdot} = \varepsilon \varepsilon_a \frac{q_{i0}}{\tau_i} q_i^{\cdot\prime}$$

The last equation of (1. 1) now yields at once

$$\frac{\partial q_i}{\partial r} = - N_i q_i^{\cdot}, \quad N_i = \frac{L}{a_0 \tau_i}, \quad i = 1, \ldots, N$$

and comparison with the formula (5. 3) gives

$$q_i^{\cdot} = - \sum_{l=1}^{N} \left(\frac{N_{il}}{N_i} \right) h_{il0} \omega_l$$

It is clear that N_{il} are the only real parameters while the numbers N_i can be eliminated altogether from the asymptotic analysis of equations.

We now use Eqs. (2. 3) and (2. 4) to derive the missing relation connecting the velocity of the mixture particles with the components of the reaction completeness vector. Let us write the increment in the M-tuply frozen and $(N - M)$-tuply equilibrium speed of sound in terms of the increments of the density, entropy, M components of the reaction completeness vector and $N - M$ components of the chemical affinity vector, as follows:

$$a_{fe0}^{(M)} a_{fe}^{(M)} = \left(\frac{\partial a_{fe}^{(M)}}{\partial \rho_0} \right)^{(M)}_{q_j, \omega_k, s} \rho_0 \rho + \left(\frac{\partial a_{fe}^{(M)}}{\partial s_0} \right)^{(M)}_{q_j, \omega_k, \rho} s_0 s +$$

$$\varepsilon_a \left[\sum_{m=1}^{M} \left(\frac{\partial a_{fe}^{(M)}}{\partial q_{m0}} \right)^{(M)}_{q_j, \omega_k, \rho, s} q_{m0} q_m + \sum_{n=M+1}^{N} \left(\frac{\partial a_{fe}^{(M)}}{\partial \omega_{n0}} \right)^{(M)}_{q_j, \omega_k, \rho, s} \frac{p_2}{q_{n0} \rho_0} \, \omega_n \right]$$

Recalling the estimates (3. 4) and 3. 6) and the relations (5. 2), we obtain the following expression with the accuracy to within ε_a^2 :

$$a_{fe}^{(M)} = (m_{fe0}^{(M)} - 1)\rho$$

Taking into account the expression (2. 2) defining the operator $L_{fe}^{(M)}$ and the relation (4. 1) which establishes the order of the difference between the velocity of the wave propagation and the M-tuply frozen and $(N - M)$-tuply equilibrium speed of sound in a quiescent medium we obtain, after simple manipulations,

$$2\left(\varepsilon m_{fe0}^{(M)} v - \varepsilon_a^2 \beta^{(M)}\right)\frac{\partial v}{\partial r} + \Delta\left[2\frac{\partial v}{\partial t} + (\nu - 1)\frac{v}{t}\right] = \tag{5, 5}$$

$$-\frac{1}{\rho_0 a_0^2}\varepsilon_a\left[\sum_{m=1}^{M} q_{m0}\left(\frac{\partial p}{\partial q_{m0}}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s}\frac{\partial q_m}{\partial r} + \frac{p_0}{\rho_0}\sum_{n=M+1}^{N}\frac{1}{q_{n0}}\left(\frac{\partial p}{\partial \omega_{n0}}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s}\frac{\partial \omega_n}{\partial r}\right]$$

Here, in accordance with formulas (3. 3), (1. 5) and (3. 1), the thermodynamic derivatives are

$$\left(\frac{\partial p}{\partial q_{m0}}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} = -\varepsilon_a p_0\left[\frac{1}{q_{m0}}\, e_{mV0} + \sum_{l=N+1}^{N}\frac{1}{q_{l0}}\left(\frac{\partial q_l}{\partial q_{m0}}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} e_{lV0}\right]$$

$$\left(\frac{\partial p}{\partial \omega_{n0}}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} = -\varepsilon_a p_0\sum_{l=M+1}^{N}\frac{1}{q_{l0}}\left(\frac{\partial q_l}{\partial \omega_{n0}}\right)^{(M)}_{q_j,\,\omega_k,\,\rho,\,s} e_{lV0}$$

In expanding the frozen speed of sound into a series, we use the density, entropy and all components of the chemical reaction completeness vector as the independent thermodynamic variables. As the result, we obtain

$$a_{f0}a_f = \left(\frac{\partial a_f}{\partial \rho_0}\right)_{q_j,\,s}\rho_0\rho + \left(\frac{\partial a_f}{\partial s_0}\right)_{q_j,\,s} s_0 s + \varepsilon_a\sum_{i=1}^{N}\left(\frac{\partial a_f}{\partial q_{i0}}\right)_{q_j,\,\rho,\,s} q_{i0}q_i$$

from which follows

$$a_f = (m_{f0} - 1)\rho$$

with the accuracy to within ε_a^2.

Using now the second expression of (4. 2) for the difference between the velocity of wave propagation and the frozen speed of sound in a quiescent state, we transform Eq. (2. 4) to the form

$$2\left(\varepsilon m_{f0} v - \varepsilon_a^2 \beta_f\right)\frac{\partial v}{\partial r} + \Delta\left[2\frac{\partial v}{\partial t} + (\nu - 1)\frac{v}{t}\right] = \tag{5, 6}$$

$$-\varepsilon_a\frac{1}{\rho_0 a_0^2}\sum_{i=1}^{N} q_{i0}\left(\frac{\partial p}{\partial q_{i0}}\right)_{q_j,\,\rho,\,s}\frac{\partial q_i}{\partial r}$$

Let us compare the values of the thermodynamic coefficients $m_{fe0}^{(M)}$ and m_{f0}. In accordance with Eq. (3. 8), their difference is of the order of ε_a^2. Turning our attention to (4.1) and (4. 2) we see that we can assume, within the present approximation, that

$$m_{fe0}^{(M)} = m_{f0} = m_0 = \frac{1}{2\rho_0^3 a_0^2}\left(\frac{\partial^2 p}{\partial V_0^2}\right)^{(M)}_{q_j,\,\omega_k,\,s} = \frac{1}{2\rho_0^3 a_0^2}\left(\frac{\partial^2 p}{\partial V_0^2}\right)_{q_j,\,s}$$

Let us now set $M = N$ in Eq. (5.5). We have $\beta^{(M)} = \beta_f$, the second sum in its right-hand part vanishes, and it becomes (5.6). A direct check confirms that Eqs. (5.5) and (5.6) coincide irrespective of the choice of the independent thermodynamic functions. The latter of the two equations is simpler and therefore preferable to be used as the missing relation connecting the particle velocity with the components of the chemical reaction completeness vector.

6. Canonical form of the chemical reaction equations. We introduce the vector

$$\mathbf{e} = (e_1, \ldots, e_N), \quad e_i = e_{iV}. \tag{6.1}$$

and two symmetric positive definite matrices

$$\begin{aligned} \mathbf{F} &= \| f_{il} \| = \| N_{il} h_{il0} \| \\ \mathbf{G} &= \| g_{il} \| = \left\| \frac{\rho_0}{p_0} q_{i0} q_{l0} \left(\frac{\partial \omega_l}{\partial q_{i0}} \right)_{q_j, V, s} \right\| \end{aligned} \tag{6.2}$$

Using the above notation we can write (5.3) and (5.4) as

$$\partial \mathbf{q} / \partial r = \mathbf{F} \boldsymbol{\omega}, \quad \boldsymbol{\omega} = \mathbf{G}\mathbf{q} - \mathbf{e}\rho = \mathbf{G}\mathbf{q} - \mathbf{e}v \tag{6.3}$$

respectively.

Transforming the components of the vectors $\mathbf{q}$ and $\boldsymbol{\omega}$, we can reduce Eq. (6.3) to their canonical form [8, 9]. Before anything else, we note that we can always choose a matrix $\mathbf{C}$ for which the product $\mathbf{F}_1 = \mathbf{C}^*\mathbf{F}\mathbf{C}$ is the unit matrix $\mathbf{E}$, where $\mathbf{C}^*$ denotes the matrix transposed to the matrix $\mathbf{C}$ [10]. Let us perform a change in variables

$$\mathbf{q}_1 = \mathbf{C}^*\mathbf{q}, \quad \boldsymbol{\omega}_1 = \mathbf{C}^{-1}\boldsymbol{\omega} \tag{6.4}$$

Here $\mathbf{C}^{-1}$ is an inverse matrix of $\mathbf{C}$. Finally, we have

$$\frac{\partial \mathbf{q}_1}{\partial r} = \mathbf{E}\boldsymbol{\omega}_1, \quad \boldsymbol{\omega}_1 = \mathbf{G}_1\mathbf{q}_1 - \mathbf{e}_1 v, \quad \mathbf{e}_1 = \mathbf{C}^{-1}\mathbf{e}, \quad \mathbf{G}_1 = \mathbf{C}^{-1}\mathbf{G}(\mathbf{C}^{-1})^* \tag{6.5}$$

Since the matrix $\mathbf{G}$ is symmetric, $\mathbf{G}_1$ is also symmetric. Therefore an unitary matrix $\mathbf{U}$ exists such that the product $\mathbf{G}_2 = \mathbf{U}^*\mathbf{G}_1\mathbf{U} = \mathbf{U}^{-1}\mathbf{G}_1\mathbf{U}$ is a diagonal matrix $\mathbf{D}$. Let us carry out another change of variables

$$\mathbf{q}_2 = \mathbf{U}^{-1}\mathbf{q}_1, \quad \boldsymbol{\omega}_2 = \mathbf{U}^{-1}\boldsymbol{\omega}_1 \tag{6.6}$$

The first equation of (6.5) is invariant with respect to such transformations, therefore we have

$$\frac{\partial \mathbf{q}_2}{\partial r} = \mathbf{E}\boldsymbol{\omega}_2, \quad \boldsymbol{\omega}_2 = \mathbf{D}\mathbf{q}_2 - \mathbf{e}_2 v, \quad \mathbf{e}_2 = \mathbf{U}^{-1}\mathbf{e}_1, \quad \mathbf{D} = \mathbf{U}^{-1}\mathbf{G}_1\mathbf{U} \tag{6.7}$$

Let us compute the diagonal elements of the matrix $\mathbf{D} = \| d_{il} \|$. They are equal to the eigenvalues of the matrix $\mathbf{G}_1$ since the transformation $\mathbf{U}^{-1}\mathbf{G}_1\mathbf{U}$ leaves them unaffected [10]. Further, the equality $\mathbf{C}^*\mathbf{F}\mathbf{C} = \mathbf{E}$ means that $\mathbf{C}^*\mathbf{F} = \mathbf{C}^{-1}$ and $\mathbf{G}_1 = \mathbf{C}^*\mathbf{F}\mathbf{G}\,(\mathbf{C}^*)^{-1}$. We see that the diagonal elements of the matrix $\mathbf{D}$ coincide with the eigenvalues of the product $\mathbf{F}\mathbf{G}$ of the matrices $\mathbf{F}$ and $\mathbf{G}$ given by (6.2).

Using different arguments we arrive at the relations $\mathbf{F}\mathbf{C} = (\mathbf{C}^*)^{-1} = (\mathbf{C}^{-1})^*$, and $\mathbf{G}_1 = \mathbf{C}^{-1}\mathbf{G}\mathbf{F}\mathbf{C}$. We can also assert that the diagonal elements of the matrix $\mathbf{D}$ are equal to the eigenvalues of the matrix $\mathbf{G}\mathbf{F}$. Since the matrices $\mathbf{F}$ and $\mathbf{G}$ are both symmetric, we have $\mathbf{G}\mathbf{F} = (\mathbf{F}\mathbf{G})^*$, i.e. eigenvalues of the matrices $\mathbf{F}\mathbf{G}$ and $\mathbf{G}\mathbf{F}$ are the same. Let us denote them by $\lambda_1, \ldots, \lambda_N$, then $d_{il} = 0$ $(i \neq l)$ and $d_{ii} = \lambda_i > 0$. As regards the vector $\mathbf{e}_2$, combining the third equations of the system (6.5) and (6.7) yields $\mathbf{e}_2 = \mathbf{U}^{-1}\mathbf{G}^{-1}\mathbf{e}$ and the vector $\mathbf{e}$ is defined in terms of the known thermodynamic derivatives $e_i \mathbf{v}_*$ using the formulas (6.1).

It can easily be shown that the scalar product $\mathbf{e}\partial\mathbf{q} / \partial r$ is invariant under the linear transformations introduced above. From this it follows that Eq. (5.6) finally assumes the form

$$2(\varepsilon m_0 v - \varepsilon_a{}^2\beta_f)\frac{\partial v}{\partial r} + \Delta\left[2\frac{\partial v}{\partial t} + (\nu_* - 1)\frac{v}{t}\right] = \varepsilon_a{}^2\frac{p_0}{\rho_0 a_0{}^2}\mathbf{e}_2\frac{\partial \mathbf{q}_2}{\partial r}$$

and, together with Eqs. (6.7) it forms a closed system of the order equal to the number of the relaxation processes plus one.

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ON THE APPLICATION OF CERTAIN GENERALIZATIONS OF THE AREA THEOREM IN SYSTEMS WITH ROLLING OF RIGID BODIES

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Among the attempts to extend the applicability conditions of the general theorems of dynamics, a prominent position is occupied by several generalizations of the area theorem proposed by Chaplygin and successfully applied by him to solving a number of problems on the rolling of spheres [1, 2]. Further general-